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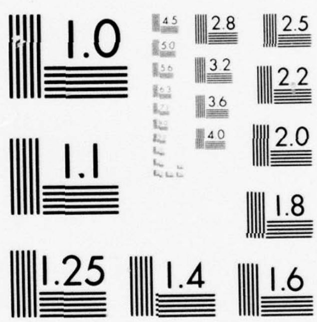

REPORT SAMSO-TR-77-43

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# On the Spectra of the Orr-Sommerfeld Equation

Vehicle Engineering Division
Engineering Science Operations
The Aerospace Corporation
El Segundo, Calif. 90245

31 January 1977



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Edward G Taylor

Edward G. Taylor, Capt. USAF
Project Engineer

FOR THE COMMANDER

Donald A Dowler

Donald A. Dowler, Col. USAF
Director, Systems Technology
Deputy for Reentry Systems

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It is shown that the continuous spectra of both the temporal and spatial eigenvalues of the Orr-Sommerfeld equation for a Blasius flow can be reinterpreted as discrete spectra of the heat-conduction and diffraction types, respectively. Most of the general properties of the discrete spectrum of the Orr-Sommerfeld equation discovered by Mack can be understood on the basis of a simple model equation.

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## 1. INTRODUCTION

The recent studies by Jordinson[1] and Mack[2] of the temporal eigenvalues of the Blasius boundary layer have provided new insight into their properties and raised a number of questions about their role in initial value and other problems. Mack found that for a given Reynolds number and wave number there exists a finite number of discrete eigenvalues; as the Reynolds number is increased, additional eigenvalues spring from the continuous spectrum. Mack discusses the properties of the continuous spectrum, which have also been considered by Grosch and Salwen (unpublished). These studies show that the eigenfunctions of the continuous spectrum vary sinusoidally as y goes to infinity. Thus these functions are bounded but are not zero in this limit. Our purpose in this paper is to offer an interpretation of the continuous spectrum and an explanation of the properties of the discrete spectrum. We shall also make a parallel study of the spatial eigenvalues.

## 2. A MODEL EQUATION

Consider the differential equation

$$R^{-1}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = U(y)\,\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial t} \qquad (1)$$

with $\phi = 0$ when $y = 0$, $\phi \to 0$ as $y \to \infty$. Here $R^2$ is a (constant) Reynolds number and $U(y)$ a given velocity distribution taken to be $U = 0$ when $y < 1$, $U = 1$ when $y > 1$.

We are primarily concerned in this paper with the spectral properties of the Orr-Sommerfeld equation with a Blasius velocity profile. The model equation, Eq. (1), is introduced because its spectral properties are generally similar to those of the Orr-Sommerfeld equation and as a result we are better able to understand the latter. A particular advantage of the model equation is that the eigenvalue problem can be reduced to a simple transcendental equation. A potential problem with the model equation arises from the use of a discontinuous U, a profile which might introduce spurious modes into the spectrum. Mack[2] has found that the discrete spectrum of the Orr-Sommerfeld equation changes significantly when the velocity profiles are changed from analytic to nonanalytic functions. As will be seen, however, the model equation with the nonanalytic profile has asymptotic character and spectral properties similar to the Orr-Sommerfeld problem.

For temporal eigenvalues we write

$$\phi = \Phi(y) \exp(i\alpha x - i\alpha c_m t) \tag{2}$$

where $\alpha$ is real, $c_m$ is the eigenvalue to be found and $\Phi$, $\Phi'$ are taken to be continuous at $y = 1$. We shall show that a discrete spectrum exists with eigenfunctions which are zero at infinity as well as a continuous spectrum with bounded eigenfunctions at infinity.

Consider first the discrete eigenvalue problem which may be reduced to the solution of

$$\theta_2 = - \theta_1 \coth \theta_1 \tag{3}$$

where

$$\theta_1^2 = \alpha^2 - i\alpha c_m R, \qquad \theta_2^2 = \alpha^2 + i\alpha R(1-c_m) \tag{4}$$

and $\mathrm{Re}\,\theta_2 > 0$. When $R \gg 1$ it is easy to show that

$$c_m = \frac{-i}{\alpha R}(\alpha^2 + n^2\pi^2) + \frac{n^2\pi^2\sqrt{2}}{(\alpha R)^{3/2}}(1+i) + \ldots \tag{5}$$

where n is an integer. Thus in the limit $R \to \infty$ there are an infinite number of discrete eigenvalues. It may be shown that there are no discrete eigenvalues for sufficiently small values of R and that $\mathrm{Re}\, c_m \neq 0$, $\mathrm{Im}\, c_m \neq 0$. Denoting the $n^{th}$ eigenvalue by $c_{mn}$, we now consider its variation with R, assuming it to be continuous. By analogy with Mack's[2] numerical work, we expect the discrete eigenvalues to terminate in the continuous spectrum and therefore look for a value $R_n$ of R such that $\mathrm{Re}\, c_{mn}$ is unity (as will be seen subsequently, $\mathrm{Re}\, c_m = 1$ is in the continuous spectrum). At this value of R we set

$$c_{mn} = 1 - \frac{i\alpha\nu}{R_n}, \quad \mu^2 = (\nu-1)\,\alpha^2, \quad \theta_1 = a - ib, \quad \theta_2 = \epsilon + i\mu \tag{6}$$

where $\mu$, $\nu$, a, b, $\epsilon$ are all real and positive. The quantity $\epsilon$ is arbitrarily small and can be neglected (we consider points arbitrarily close to the continuous spectrum). With further manipulation, we obtain

$$b^2 = \mu^2 + a^2, \qquad ab = \alpha R_n/2 \tag{7}$$

It may readily be shown that

$$b = a\cosh(2a), \quad a = -b\cos(2b), \quad \sin(2b) = -\tanh(2a), \quad \mu = a\sinh(2a); \tag{8}$$

again there are an infinite number of solutions of which the smallest has

$$a = 0.815, \; b = 2.163, \; \alpha R_1 = 3.526, \; c_{m_1} = 1 - i(0.284\alpha^2 + 1.138)$$

and, when n is a large integer

$$b \approx (n - \tfrac{1}{4})\pi, \;\; a \approx \log[(n - \tfrac{1}{4})\pi]/2, \;\; \alpha R_n = 2ab, \;\; c_{mn} \approx 1 - \frac{i\alpha^3 R_n^2}{\log(2\alpha R_n)} \tag{9}$$

Further, it may be shown, and is in fact implicit in Eq. (8), that there are no solutions in which $\mu$ is either negative or imaginary. In Fig. 1 we display the variation of Re $c_{mn}$ and Im $c_{mn}$ for two of the principal eigenvalues.

The inference is clear. When R takes an infinite value there is also an infinite number of discrete eigenvalues. At finite values of R there is only a finite number, and for each there is a critical value $R_n$ of R at which Re $c_{mn}$ reaches unity; at smaller values of R this eigenvalue is no longer physically meaningful. An analytic continuation of the solution of Eq. (3) may be found when $R < R_n$, corresponding to Re $c_m > 1$, but then Re $\theta_2 < 0$ and therefore we have a problem in which it is desirable to remove the exponentially <u>decaying</u> solution as $y \to \infty$. It is natural to ask whether there are any other solutions in which Re $c_m > 1$ and Re $\theta_2 < 0$; the answer appears to be no. Certainly if such solutions intersect the continuous spectrum, then as Re $c_m \to 1+$ we have $\theta_2 = i\mu$ with $\mu < 0$, and we know that no such solutions exist.

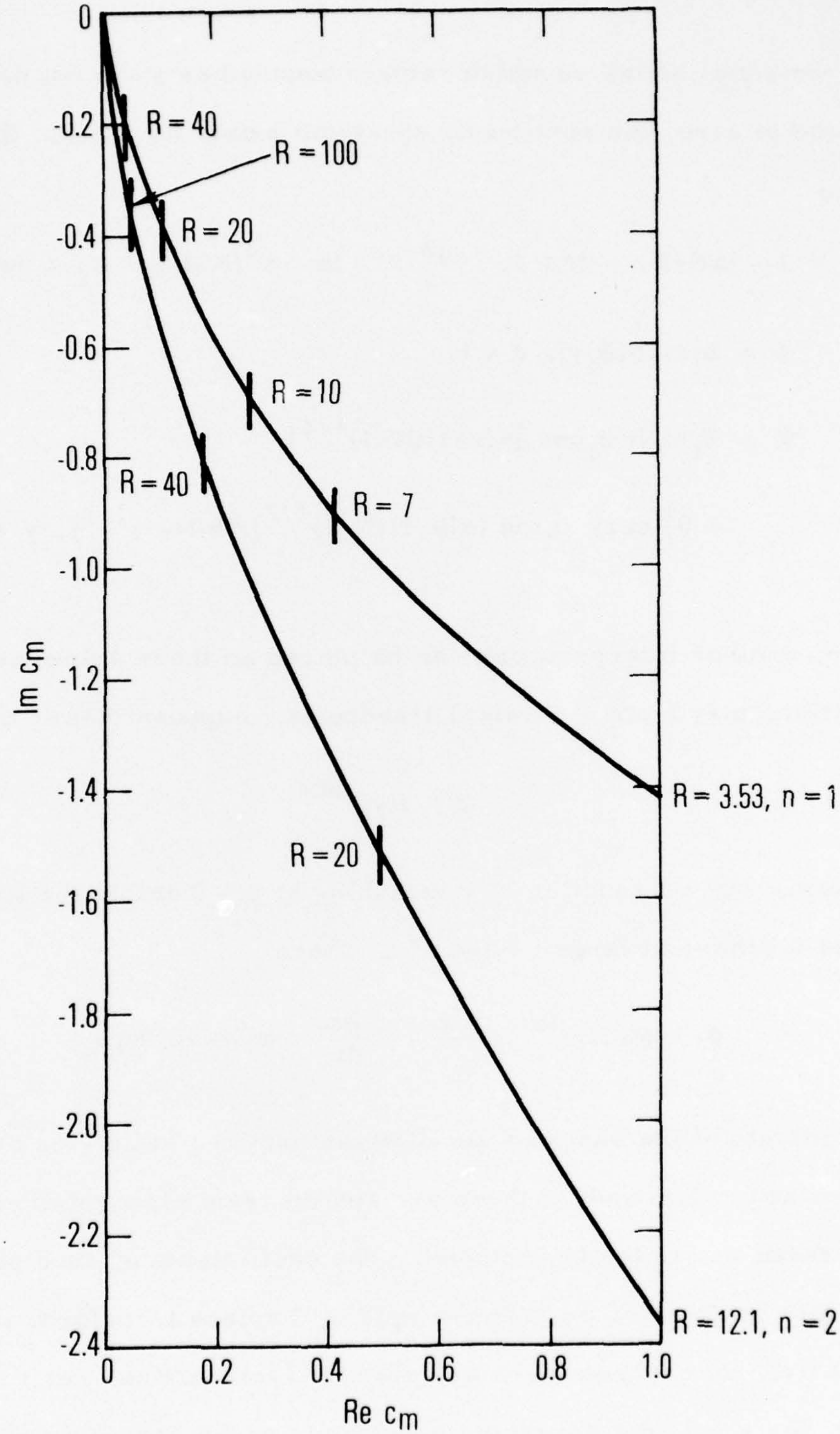


Figure 1. Variation of Re $c_m$ and Im $c_m$ with R for the first two discrete temporal eigenvalues of the model equation when $\alpha = 1$. They branch off the continuous spectrum when R = 3.53 and 12.1, respectively.

If we admit solutions which remain bounded as $y \to \infty$ but do not necessarily tend to zero, the continuous spectrum exists for all R. These solutions are

$$c_m = 1 - i\alpha N/R, \quad N > 1, \quad \theta_1^2 = -i\alpha - \alpha^2(N-1), \quad \theta_2 = i\alpha(N-1)^{1/2} \tag{10}$$

$$\Phi = \theta_1 \sinh(\theta_1 y), \; y < 1,$$

$$\Phi = \theta_1 \sinh \theta_1 \cos[\alpha(y-1)(N-1)^{1/2}]$$

$$+ \theta_1^2 \cosh \theta_1 \sin[\alpha(y-1)(N-1)^{1/2}]/[\alpha(N-1)^{1/2}], \; y > 1 \tag{11}$$

However, another interpretation can be placed on these solutions which is more satisfactory from a physical standpoint. Suppose that at time $t = 0$

$$\phi = f(y)e^{i\alpha x} \tag{12}$$

where $f(y)$ is a given function of y vanishing at $y = 0$ and in the limit as $y \to \infty$. How does $\Phi$ behave at large values of t, where

$$\phi = \Phi(y,t)e^{i\alpha x}, \quad \frac{\partial^2 \Phi}{\partial y^2} - R\frac{\partial \Phi}{\partial t} = (\alpha^2 + i\alpha UR)\Phi \tag{13}$$

It then consists of the sum of a set of eigenfunctions satisfying null boundary conditions at $y = 0, \infty$ and, if there are any discrete eigensolutions, i.e., if $R > R_1$, these must also be included. The coefficients of such eigenfunctions may readily be determined, for example by Laplace transform analysis, in terms of $f(y)$. In addition there are always eigenfunctions derived from the fact that, for $y > 1$, the governing equation is of the heat-conduction type.

Therefore, we may expect that at large values of t the solution of Eq. (13) in $y > 1$ will include an expression of the form

$$\Phi_1 = \exp(-\alpha^2 tR^{-1} - i\alpha t)\sum_m t^{-m} F_m(yR^{1/2}t^{-1/2}) \qquad (14)$$

due to this cause. Here $F_m$ satisfies Hermite's equation, e.g., $F_o = \exp(-y^2R/4t)$ and $F_m \to 0$ as $y \to \infty$. At $y = 1$ this solution and its first derivative with respect to y must be continuous with a solution of Eq. (13) in $0 < y < 1$. This solution may be expressed in the form

$$\Phi_o = \exp(-\alpha^2 tR^{-1} - i\alpha t)\sum_p t^{-p} f_p(y) \qquad (15)$$

where $f_p$ is a complex exponential function satisfying $f_p(0) = 0$. By matching Eqs. (14) and (15), across $y = 1$ we establish immediately that $F_m(0) = 0$ for them conjointly to describe the eigenfunctions as $t \to \infty$. This condition in turn implies that m must be a positive integer and $p = m + 1/2$.

Thus, we see that at a finite value of R the eigenfunctions consist mainly of an infinite set of functions which decay exponentially in time, decay exponentially with y outside a region of thickness $(t/R)^{1/2}$, and have a boundary-layer structure in $0 < y < 1$. Insofar as the effect on the external region $y > 1$ is concerned, these are the most significant because their thickness grows with t. The phase velocity of these solutions is given by

$c_m = 1 - i\alpha/R$. Now each of the functions $F_m$ can be expressed in the form

$$\int_{-\infty}^{\infty} g_m(\theta) \exp\left\{-i\theta y - \theta^2 t R^{-1}\right\} d\theta \tag{16}$$

and so may be regarded as a sum of the eigenfunctions of the continuous spectrum (Friedman[3] has made such an identification for the continuous spectrum of the membrane equation). In addition to the parabolic eigen-solutions there is, for $R > R_1$, another discrete set of the form of Eq. (2) in which $0 < \mathrm{Re}\, c_m < 1$. As R increases, this set is augmented at discrete values of R by new eigensolutions, which may be regarded as springing from the continuous spectrum whenever Eq. (3) is satisfied, so that at these points Eq. (11) reduces to

$$\phi = \theta_1 \sinh \theta_1 \exp\left\{-i\alpha(y-1)\sqrt{N-1}\right\} \text{ in } y > 1 \tag{17}$$

## 3. TEMPORAL SPECTRUM OF THE BLASIUS BOUNDARY LAYER

The Orr-Sommerfeld problem may be formulated as follows. Consider the equation for the small perturbations $\psi$ of the streamfunction for a steady shear flow defined by a velocity U(y) in the x direction. Disregarding the fact that the steady flow does not satisfy the Navier-Stokes equations, we have

$$\frac{\partial}{\partial t}(\nabla^2\psi) + U\frac{\partial}{\partial x}(\nabla^2\psi) - U''(y)\frac{\partial\psi}{\partial x} = R^{-1}\nabla^4\psi \tag{18}$$

where $R^2$ is the Reynolds number. For the Blasius boundary layer $U(0) = 0$ and $U(\infty) = 1$. Further, $xx_o R^{-1}$ measures distance along the plate from the center of the disturbance, itself a distance $x_o$ from the leading edge of the plate; $yx_o R^{-1}$ measures distance normal to the plate; $tx_o R^{-1}$ measures time; and $R^2$ is based upon $x_o$.

The boundary conditions satisfied by $\psi$ may be taken as

$$\psi = \frac{\partial\psi}{\partial y} = 0 \quad \text{at } y = 0 \text{ and } \psi \to 0 \text{ as } y \to \infty \tag{19}$$

for all x and positive t. The assumptions leading to Eq. (18) are valid when $R \gg 1$, but here we shall consider the properties of the solution of Eqs. (18) and (19) for all R.

The standard method for finding the temporal eigenvalues of $\psi$ is to write

$$\psi = e^{i\alpha(x-ct)}\Psi(y) \tag{20}$$

with $\alpha$ being real, whereupon $\Psi$ satisfies the Orr-Sommerfeld equation with c as eigenvalue. This is a fourth-order equation and, as $y \to \infty$, has a richer asymptotic structure than our model equation which is only second order. In addition to the term $e^{-\theta_2 y}$ [cf. Eq. (4)], $\Psi$ can include a term $\propto e^{-\alpha y}$. The model equation, if $y > 1$, is actually the same as that for $\Psi$ if $y \gg 1$, so it is not surprising that in spite of its relative simplicity, practically all the general properties of $\Phi$ and $c_m$ have their parallel in those of $\Psi$ and c as elucidated by Mack.[2] First of all there is a continuous spectrum given by

$$c = 1 - \frac{i}{\alpha R}(p^2 + \alpha^2), \qquad p \text{ real,} \tag{21}$$

for which $\Psi$ is merely bounded as $y \to \infty$.

This spectrum may also be interpreted in terms of the discrete spectrum of the heat conduction equation. We write

$$\psi = e^{i\alpha(x-t) - \alpha^2 t/R}\,\tilde{\Psi}(y,t) \tag{22}$$

and set

$$\tilde{\Psi} = t^{-m-1/2}\sum_{n=0}^{\infty} \Psi_n(y) t^{-p_n} \qquad \text{if } y \sim 1, \tag{23}$$

where m, $p_n$ are real, $p_o = 0$ and $p_n > p_{n-1}$. Then $\Psi_o$ satisfies

$$\Psi_o''' - \alpha^2 \Psi_o'' + i\alpha R U''(y)\Psi_o = 0 \tag{24}$$

and has a solution in which $\Psi_o(0) = \Psi_o'(0) = 0$ and

$$\Psi_o \sim A_o e^{-\alpha y} + B_o y + C_o \qquad \text{as } y \to \infty . \tag{25}$$

There is no numerical evidence that the constant $B_o$ is ever zero. When $y \sim t^{1/2}$ and t is large, we write

$$\tilde{\Psi} = t^{-m}\sum_{n=0}^{\infty}\left[\tilde{\Psi}_n(y)t^{-p_n - 1/2} + F_n(yR^{1/2}t^{-1/2})t^{-q_n}\right] \tag{26}$$

where $q_n > q_{n-1}$, $\tilde{\Psi}_o'' - \alpha^2\tilde{\Psi}_o = 0$ and $F_o$ is defined in Eq. (14). A match between Eqs. (25) and (26) is only possible if $\tilde{\Psi}_o = A_o e^{-\alpha y}$, $F_o(0) = 0$, $F'(0) = R^{-1/2} B_o$, and $q_o = 0$. Since $F_o \to 0$ as $y/t^{1/2} \to \infty$, it follows that m is an integer. Thus, as with the model equation there is a discrete spectrum of the heat-conduction type, each member of which may be interpreted as an integral over the continuous spectrum.

The eigenfunctions [Eqs. (23), (26)] should, from a strict mathematical point of view, be regarded as the central set of eigenfunctions. In addition, for given $\alpha$, there is a critical value $R_o$ of R beyond which a second, discrete set springs from the continuous spectrum. This behavior is similar to that of the model equation. The results of the two searches performed by Mack[2] and the authors suggests that at the intersection points of the continuous and discrete spectrum $\Psi$, which oscillates finitely when $y \gg 1$, has a form similar to Eq. (17), especially in regard to the sign of the argument of the exponential. No solutions were found in which this sign was positive. This suggests that there is no branching from the continuous spectrum leading to eigenvalues in which Re $c > 1$. The value of $R_o^2$ was shown by Mack to be 15 when $\alpha = 0.179$ and we found it to be 25 when $\alpha = 0.115$. As R increases, more members of this second set appear and the total number tends to infinity with R. The properties of these eigenvalues appear, with one exception, to be consistent with those of the model equation; i.e., with Re $c \to 0+$, Im $c \to 0-$ as $R \to \infty$. The exception is the inviscid eigenvalue, which Mack[2] has labelled 3 when $R \sim 10^2$, as shown in Fig. 3 of his paper. For this one c tends to a finite limit. These two types of limiting behavior are consistent with the two classes of asymptotic solutions described by Lin.[4] A further special feature of these eigenvalues distinguishing them from $c_m$, and of crucial importance, is that their real and imaginary parts are not necessarily monotonic; it is possible for Im $c > 0$.

## 4. THE CONTINUOUS SPECTRUM OF SPATIAL EIGENVALUES

The spatial eigenvalues, obtained by requiring R and $\omega = \alpha c$ to be real quantities and looking for (possibly complex) values of $\alpha$ which permit non-trivial solutions of the Orr-Sommerfeld equation, are also of interest and have been studied by Gaster and Jordinson[5] in connection with the evolution of initial isolated disturbances in the boundary layer. The neutral curves, on which Im $\alpha = 0$, are identical to the neutral curves of the temporal eigenvalues, but we do not have such a detailed knowledge of the structure of $\alpha$ elsewhere in the complex plane as that provided by Mack[2] in the temporal case. We can see from Eq. (4), however, that there is a continuous spectrum of eigenvalues which are obtained by setting $\theta_2^2 = -p^2$ and requiring p to be real. The corresponding values of $\alpha$ are defined by

$$\alpha^2 + i\alpha R = -p^2 + i\omega R \qquad (27)$$

This spectrum may also be reinterpreted in this instance in terms of the spectrum of the diffraction equation. We write

$$\psi = e^{-i\omega t}\,\chi(x, y) \qquad (28)$$

and, when $y \sim 1$, $x \gg 1$, expand $\chi$ in the form

$$\chi \sim e^{i\alpha_1 x} \sum_{n=0}^{\infty} \chi_n(y) x^{-s_n} \tag{29}$$

where $2\alpha_1 = -iR + i(R^2 - 4i\omega R)^{1/2}$, $\chi_n$ is a function of y only, $s_n$ is a monotonic increasing sequence of real numbers and $\chi_o$ satisfies

$$R^{-1}(\chi_o'' - \alpha_1^2 \chi_o)'' = i(U-1)\alpha_1(\chi_o'' - \alpha_1^2 \chi_o) - i\alpha_1 U_o'' \chi_o, \tag{30}$$

together with the boundary conditions $\chi_o(0) = \chi_o'(0) = 0$. We now forbid $\chi_o$ to grow exponentially with y as $y \to \infty$, and then when $y \gg 1$

$$\chi_o \approx D_o e^{-\alpha_1 y} + E_o y + F_o \tag{31}$$

where $D_o, E_o, F_o$ are constants of which $E_o$ can be expected to be non-zero. When $y, x \gg 1$ we write

$$\chi = \sum_{n=0}^{\infty} \left[ \tilde{\chi}_n(y) x^{-t_n} e^{i\alpha_1 x} + e^{1/2\, Rx} G_n(r, \theta) \right] \tag{32}$$

where $t_n$ is monotone increasing, $\tilde{\chi}_o'' = \alpha_1^2 \tilde{\chi}_o$, $r \cos\theta = x$, and $r \sin\theta = y$. The differential equation satisfied by $G_o$ is

$$\frac{\partial^2 G_o}{\partial r^2} + \frac{1}{r}\frac{\partial G_o}{\partial r} + \frac{1}{r^2}\frac{\partial^2 G_o}{\partial \theta^2} = \left(\frac{R^2}{4} - i\omega R\right) G_o \tag{33}$$

and since $G_n \to 0$ as $r \to \infty$

$$G_o = \sum_m (L_m \cos m\theta + M_m \sin m\theta) K_m \left[ \left( \frac{R^2}{4} - i\omega R \right)^{1/2} r \right] \quad (34)$$

where $K_m$ is a Bessel function of the second kind. The matching of Eq. (29) as $y \to \infty$ with Eq. (32) as $y/x \to 0$ requires that

$$\tilde{\chi}_o = D_o e^{-\alpha_1 y}, \; L_m = 0, \; mM_m = E_o \quad (35)$$

and $s_n$ is an integer. At this stage m remains indeterminate.

A parallel pair of expansions can be written if x is large and negative except that $2\alpha_1$ must be replaced by $2\alpha_2 = -iR - i(R^2 - 4i\omega R)^{1/2}$ and results similar to Eq. (35) are recovered. (Recall that negative x refers to stations upstream of $x_o$ and not of the leading edge.) It may be expected that the expansions for $x \gg 1$ and $x \ll -1$ when $y \gg 1$ are analytic continuations of one another. Since the expressions for $G_o$ in Eq. (32) hold for all $\theta$ provided $r \gg 1$, we would then infer that $\sin m\pi = 0$ also, in which case m is an integer and the resulting discrete spectrum may be interpreted as an integration over the continuous spectrum defined by Eq. (27).

The model equation, Eq. (1), also has a continuous spectrum of eigenvalues, defined by Eq. (27), which may be reinterpreted in terms of the spectrum of the diffraction equation. Further, it has a readily identifiable family of discrete eigenvalues obtained by requiring R and $\alpha c_m = \omega$ to be real and by solving Eq. (3) for $\alpha$ subject to the condition Re $\theta_2 > 0$. Denoting a typical member of this family by $\alpha_M$, we may easily show that when $R \gg 1$

$$\alpha_M^2 = i\omega R + n^2\pi^2 + O(R^{-3/4}) \tag{36}$$

where n is an integer. The two solutions are presumably relevant to the regions $\pm x \gg 1$ and correspond to waves propagating upstream and downstream. When R is finite the values of $\alpha_M$ must be found numerically. In Figs. 2 and 3 we display the variations of $\alpha_M$ with R for the first two eigenvalues; the wave velocity is positive in Fig. 2 and negative in Fig. 3. These eigenvalues terminate as R decreases when Re $\theta_2 = 0$; when the waves propagate downstream this occurs at R = 6.82 and 14.3, respectively, for the first two of them. For negative wave velocity the terminal values of R are changed to 1.36 and 4.17.

Thus it follows that the spectrum of Eq. (1) consists only of a continuous family if R < 1.36, which may be interpreted as a discrete spectrum using diffraction theory. At R = 1.36 a branching occurs and for R > 1.36 there is another eigenvalue discrete from this family in which Im $\alpha_M < 0$. At R = 4.17 another eigenvalue branches off so that for R > 4.17 there are two discrete eigenvalues, each with Im $\alpha_M < 0$, and so on. We infer that for general R > 6.82 the spectrum of $\alpha_M$ consists of a continuous family defined by Eq. (27) together with a finite number of discrete values, some of which have Im $\alpha_M > 0$ and some Im $\alpha_M < 0$. The number in each set is roughly proportional to R when R is large.

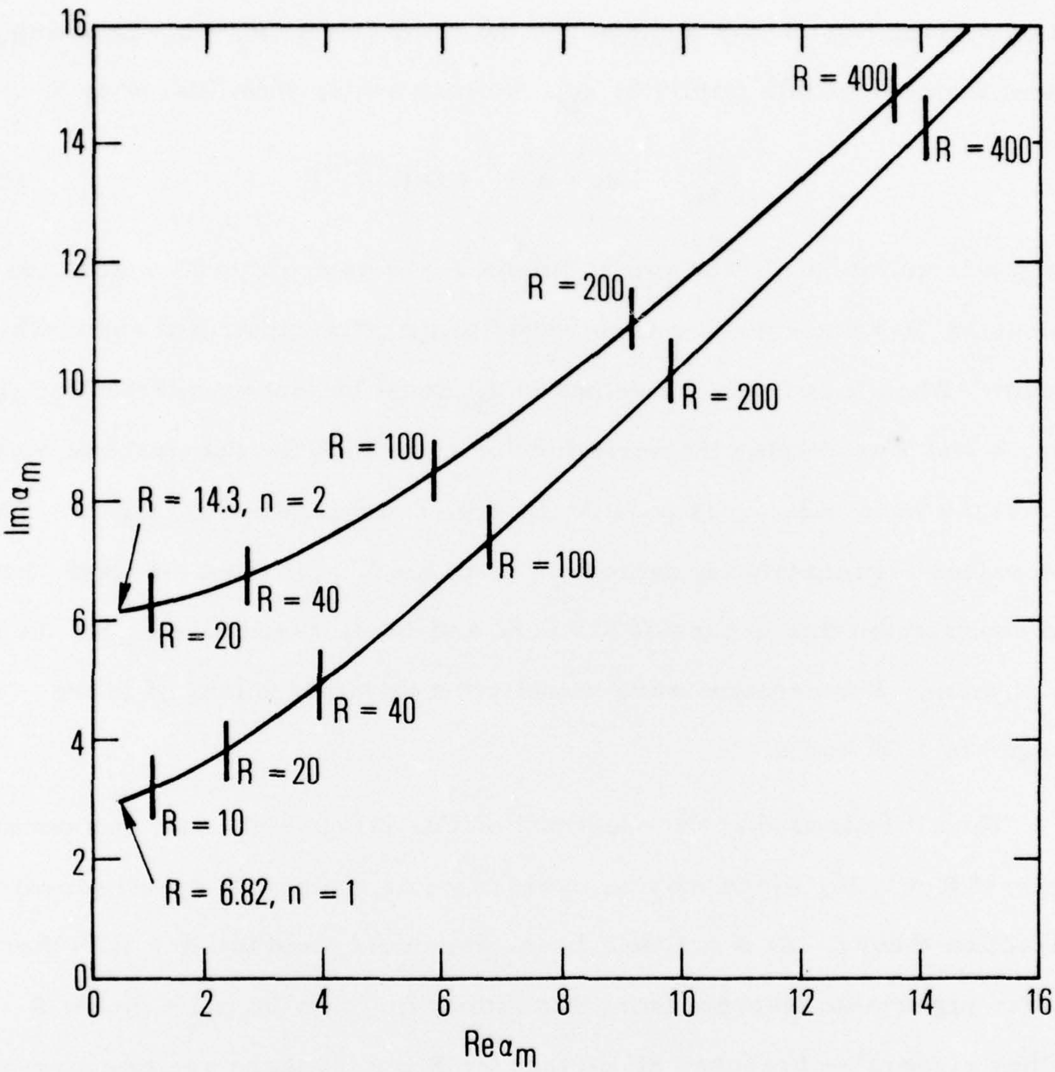


Figure 2. Variation of Re $\alpha_m$ and Im $\alpha_m$ with R corresponding to waves propagating downstream for the first two discrete spatial eigenvalues of the model equation when $\omega = 1$. They branch off the continuous spectrum when R = 6.82 and 14.3, respectively.

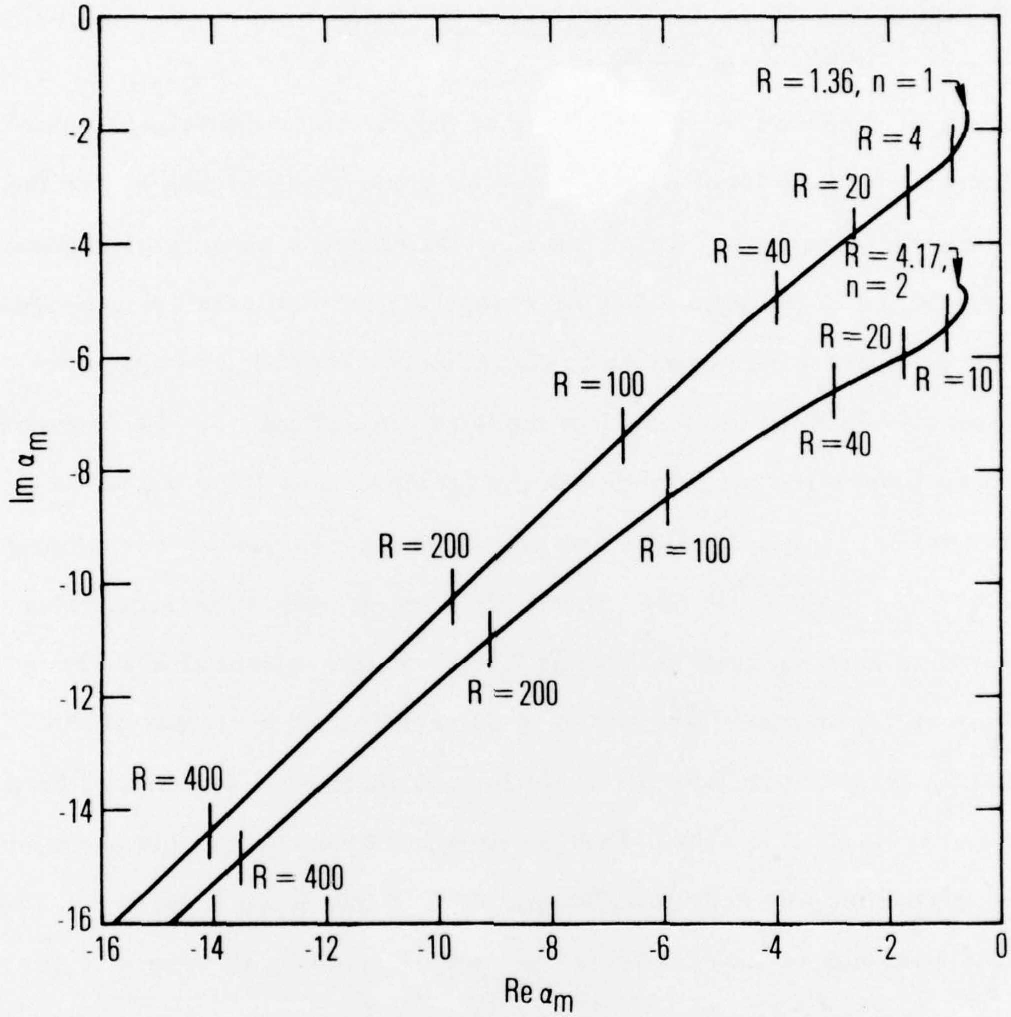


Figure 3. Variation of Re $\alpha_m$ and Im $\alpha_m$ with R corresponding to waves propagating upstream for the first two discrete spatial eigenvalues of the model equation when $\omega = 1$. They branch off the continuous spectrum when R = 1.36 and 4.17, respectively.

## 5. THE DISCRETE SPATIAL SPECTRUM OF THE BLASIUS BOUNDARY LAYER

There are at present no calculations of the discrete spatial spectrum[6] for the Blasius boundary layer as extensive as those given by Mack[2] for the discrete temporal spectrum. Thus, we have developed a numerical method (which is described in the Appendix) for computing the discrete spatial eigenvalues. Since $\mathrm{Re}\,\alpha > 0$ corresponds to disturbances traveling downstream and is of greater physical interest than those in which $\mathrm{Re}\,\alpha < 0$, the numerical work has been primarily concerned with the former. For $\mathrm{Re}\,\alpha > 0$ we have obtained the variation with R of the two principal eigenvalues corresponding to $\omega = 0.0397$, and the results are depicted in Fig. 4. The first eigenvalue springs from the continuous spectrum at $R = 10.5$; the second at $R = 265$. The behavior of the primary eigenvalue is quite different from that of the model equation (Fig. 2); neither $\mathrm{Re}\,\alpha$ nor $\mathrm{Im}\,\alpha$ is monotonic and indeed $\mathrm{Im}\,\alpha$ changes sign for $451 < R < 3360$. Furthermore, it appears that the primary eigenvalue is approaching a limit as $R \to \infty$; we find this limit to be $\alpha \approx 0.14 + 0.005i$. The behavior of the first discrete spatial mode is similar to that observed by Mack[2] for the first discrete temporal mode; the behavior for $R \to \infty$ is in agreement with Lin's[4] work. The physically most important property of Eq. (18) is the possibility of instability over a finite range of R and is manifested in the unique behavior of the first mode. The second mode in Fig. 4 and presumably the higher modes have $\mathrm{Re}\,\alpha$ and $\mathrm{Im}\,\alpha$ monotonic with R and are very similar to the model (Fig. 2).

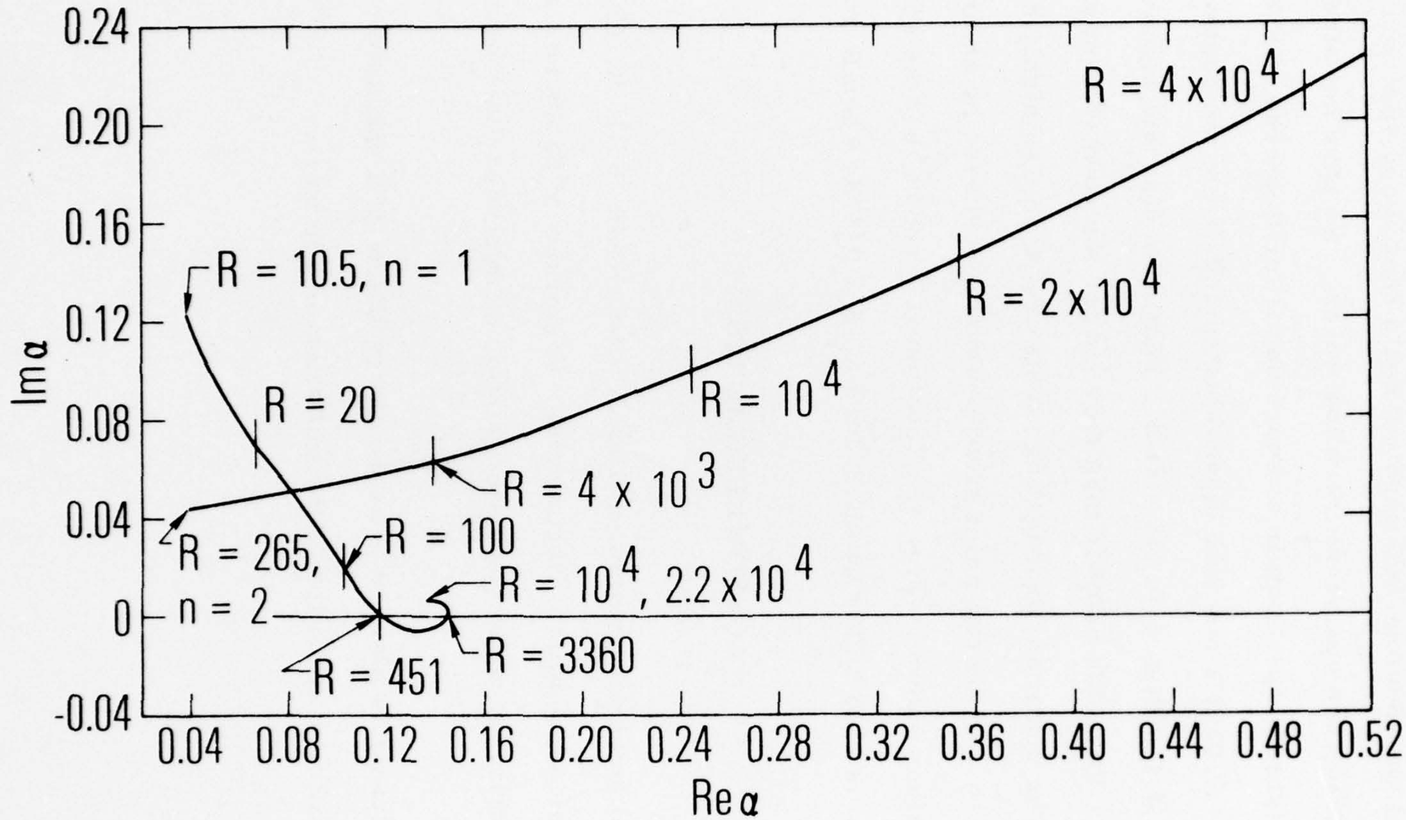


Figure 4. Variation of Re $\alpha$ and Im $\alpha$ with R corresponding to waves propagating downstream for the first two discrete spatial eigenvalues of the Orr-Sommerfeld equation when $\omega = 0.0397$. They branch off the continuous spectrum when R = 10.5 and 265, respectively.

The spatial Orr-Sommerfeld problem has a continuous spectrum with $\mathrm{Re}\,\alpha < 0$ which describes upstream traveling waves. We have not studied the corresponding discrete spectrum in as much detail for this case, but have found that for $\omega = 0.0397$ a discrete eigenvalue springs from the continuous spectrum at $R = 2.1$ with $\alpha = -0.035 - 2.2i$. Thus, the discrete eigenvalues appear at a lower Reynolds number than they did for the downstream propagating waves. The upstream propagating waves are of course highly damped. The quantity $\mathrm{Im}\, -\alpha$ for the one point given previously exceeds by an order of magnitude the values shown in Fig. 4. Equation (27) shows that the continuous spectrum is also highly damped; $\mathrm{Im}\,\alpha = -R + 0(1/R)$ when $R \gg 1$ and $p = 0(1)$.

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## APPENDIX: DESCRIPTION OF THE NUMERICAL SCHEME

To obtain the results presented herein it was necessary to solve the Orr-Sommerfeld equation numerically for selected discrete and continuous eigenfunctions and eigenvalues. The numerical method is similar to one used by Orszag[7] in that the dependent variables are represented by an appropriate series of Chebyshev polynomials. Then the discrete eigenfunctions and eigenvalues may be found by application of the QR algorithm due to Francis[8], and the continuous solutions may be found by solving a system of simultaneous linear equations.

In the present problem the domain of y is semi-infinite; therefore a convenient argument for the Chebyshev polynomials is a negative exponential function rather than the usual linear function of y. Thus, we have represented a typical dependent variable in the form

$$\Psi(y) = \sum_{j=0}^{N} a_j T_j^* \left(e^{-y/y_r}\right) \tag{37}$$

where $T^*$ is the Chebyshev polynomial defined on the interval zero to one and $y_r$ is a scale factor. (The properties of Chebyshev polynomials are given, for example, by Fox and Parker[9].)

The first step in the numerical procedure is to solve the Blasius equation so it may be represented in the form of Eq. (37). This is achieved by iterating the Weyl integral formulation of the Blasius equation, as suggested by Jones and Watson.[10] With a 17-term expansion ($N = 16$) and $y_r = 4.25$ the solution for U has a maximum error of order $10^{-5}$ on the interval $0 \leq y \leq \infty$. Since N is always greater than or equal to 16 for the results presented herein, any errors in the Blasius solutions are negligible.

The Orr-Sommerfeld equation is solved by formulating the temporal eigenvalue problem with the $a_j$'s in Eq. (37) as unknowns. This results in a linear eigenvalue problem and the QR algorithm is used to solve it. (The method is similar to the one used by Orszag[7] to solve the Orr-Sommerfeld equation for plane Poiseuille flow.)

For given values of $\alpha$ and R, the eigenvalues of the Orr-Sommerfeld equation consist of a continuous spectrum and a finite number of discrete values. The numerical model has N-2 eigenvalues, some of which are spurious and some of which approximate the actual spectrum. We make two tests for the relevance of the solutions. First the value of the eigenvalue should approach a limit as N is increased; second the series Eq. (37) describing the eigenfunction must be convergent and accurate to order $10^{-3}$ or less. [That is the a's as $j \rightarrow N$ must be $0(10^{-3})$.] About a third of the eigenvalues can be excluded on the basis of the first test since they do not approach a limit. Over half the eigenvalues are very near the continuous spectrum and have $\mathrm{Re}\, c = 1 - \epsilon$ with $\epsilon$ varying between $10^{-4}$ and $10^{-1}$ in a typical case. The quantity Im c does not approach a limit with N nor does the stream function series Eq. (37) converge. We interpret these solutions as approximate representations of the continuous spectrum by the discrete numerical model but recognize that this representation can never be accurate because the function set is countable and decays exponentially while the continuous spectrum is sinusoidal at infinity. The remaining solutions represent the discrete eigensolutions of the Orr-Sommerfeld equation. In the Reynolds number range in which instability occurs, rather modest values of N will result in three-figure accuracy for the primary discrete eigensolution; both larger and smaller values of R require an increased N. For example, with $R^2 = 10^5$, $\alpha = 0.115$, $N = 16$, and $y_e = 4.25$, the maximum error in the stream function is $0(10^{-3})$ compared to unity on the semi-infinite interval. To obtain the curves of the first two spatial modes shown in Fig. 4 we have used values of N up to 50; the value of $y_e$ has also been varied to minimize N. (The optimum value of $y_e$ varies directly with some characteristic thickness of the eigensolutions and therefore decreases with R.)

Solutions to the temporal eigenvalue problem are easily obtained: R and a real value of $\alpha$ are specified, and in some cases more than one mode may be obtained from one calculation. The spatial eigenvalue problem is more complicated: R is specified and we iterate the complex value of $\alpha$ until $\omega$ has a specified real value. This procedure works well, converging in two or three iterations. However, the first one or two points of the higher modes are more difficult to obtain because there is no convenient starting point as there is with the first mode which has $\alpha$ and $\omega$ both real at the stability boundary.

The representation of the eigenfunctions as polynomials in negative exponentials is well suited to functions of the boundary-layer type. However, as the discrete eigensolutions approach the intersection point with the continuous spectrum, the thickness increases and the numerical scheme described herein fails. In order to locate these points a numerical method for generating solutions in the continuous spectrum is used. Approximate solutions of the Orr-Sommerfeld equation in the continuous spectrum have been obtained by Rogler and Reshotko[11]; however, they considered only the case $p = \alpha$ [cf. Eq. (21)] and they neglected the imaginary part of c. The solution technique used herein is similar to that of Rogler and Reshotko; that is, we subtract the oscillatory part of the solution from the dependent variable and thereby obtain an inhomogeneous version of the Orr-Sommerfeld equation.

$$\Theta(y) = \Psi(y) - C \sin py - D \cos py \tag{38}$$

[In some cases we found that accuracy was increased by also subtracting $E \exp(-\alpha y)$ from $\Psi$ in Eq. (38).] The solution procedure is as follows: Substitute Eq. (38) into the Orr-Sommerfeld equation and the boundary conditions and specify R, p, and $\alpha$ (or $\omega$). Then $\omega$ (or $\alpha$) is given by

$$\alpha^2 + p^2 = iR(\omega - \alpha) \tag{39}$$

and $\Theta$ and the constants C and D are unknowns defined by a linear system of equations (not an eigenvalue problem). We solve this system using a Chebyshev expansion of the form Eq. (37), but that is not essential.

The intersection points are those points at which

$$|C| = |D|; \ \arg D - \arg C = \pi/2 \tag{40}$$

We locate these points by generating the curve as a function of R and p upon which the first condition of Eq. (40) is satisfied. The point on the curve where both of Eqs. (40) are satisfied is an intersection point.

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